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Factored coset approach to bosonization in the context of topological backgrounds and massive fermions

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Abstract

We consider a recently proposed approach to bosonization in which the original fermionic partition function is expressed as a product of a G/G-coset model and a bosonic piece that contains the dynamics. In particular we show how the method works when topological backgrounds are taken into account. We also discuss the application of this technique to the case of massive fermions.

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In a recent paper [1], Theron et al presented an alternative approach to bosonization in two dimensions using the path integral formalism. They obtained a complete derivation of the bosonization dictionary for both the Abelian and non-Abelian cases. In the Abelian case they started with the generating functional for current-current correlation functions of free Dirac fermions in two dimensional Euclidean space:

$$Z = \int D\bar{\Psi}\ D\Psi DA \exp\{-\int d^2x\ [\bar{\Psi} i\partial\!\!\!/ \Psi - j_\mu A_\mu]\} \tag{1}$$

where $j_\mu = \bar{\Psi}\gamma_\mu\Psi$.

In a general gauge, that is without fixing the gauge, the method consists in making a gauge transformation in the fermionic variables which gives rise to a delta of conservation of the fermionic current $j_\mu = \bar{\Psi}\gamma_\mu\Psi$. Appropiately representing $\delta(\partial_\mu j_\mu)$ and making a chiral change of variables Z can be factorized in terms of a G/G-coset fermionic partition function, and a bosonic part which contains the dynamics:

$$Z = \int D\bar{\Psi}\ D\Psi\ DB_\mu e^{-S_{cf}} \int D\phi DA_\mu e^{-S_{bos}[\phi,A_\mu]} \tag{2}$$

where S_{cf} is the action of the fermionic coset $U(1)/U(1)$ model.

This is one of the main achievements of the new approach. Indeed, in the standard decoupling technique of the path integral bosonization [2], $Z[A_\mu]$ is expressed as a bosonic partition function multiplied by the vacuum to vacuum amplitude of free fermions. In the framework of Ref.[1] the fermionic factor corresponds to constrained fermions, which are dynamically trivial (in the sense that both fermionic currents are set equal to zero). Thus, the bosonizing character of the procedure becomes more apparent.

Taking this into account it is interesting to analyze the applicability of this new method to the study of other physical situations in which the bosonization procedure is known to be more involved than in the case of free massless fermions. In this work we focus our attention on two of such situations. Firstly we consider a model of fermions coupled to a vector field A_μ, allowing this field to carry a non-trivial topological charge: $\oint A_\mu dx^\mu = -2\pi N$ [3]. One of the more interesting features of this model is the existence of the so called minimal correlation functions which, being zero for trivial topology, develop non-zero values when $N \neq 0$.

Finally we go back to trivial topology and briefly show how to extend the method of [1] to the case in which fermions are massive. This is an important point since much of the pioneering work on bosonization was done in the context of massive models [4].

On the other hand, our discussion concerning this matter could also be helpful in order to use the ideas of [1] in the study of 2D statistical-mechanics models away from criticality. In this context $\bar{\Psi}\Psi$ is the energy density of the system and $m \propto (T - T_c)$.

Let us start with the generating functional introduced in Ref [3]:

$$Z = \sum_N \int D\bar{\Psi}\ D\Psi D A_\mu \exp\{-\int d^2x\ \bar{\Psi}[i\partial\!\!\!/ + A\!\!\!/]\Psi\} \tag{3}$$

with $A_\mu = A_\mu^{c(N)} + a_\mu$, where $A_\mu^{c(N)}$ is a fixed (classical) configuration with topological charge N:

$$\oint A_\mu^{c(N)} dx^\mu = -2\pi N \tag{4}$$

while a_μ stays in the topologically trivial ($N = 0$) sector.

Following the procedure developed in [1] to obtain a coset model in a general gauge, we perform a gauge trasformation

$$\Psi \to e^{i\eta(x)}\Psi \qquad\qquad \bar{\Psi} \to \bar{\Psi} e^{-i\eta(x)} \tag{5}$$

in the generating functional (3). We then have

$$Z = \sum_N \int D\bar{\Psi}\ D\Psi D A_\mu \exp\{-\int d^2x\ [\bar{\Psi} i\partial\!\!\!/ \Psi + j_\mu A_\mu + (\partial_\mu j_\mu)\eta]\} \tag{6}$$

Due to gauge invariance, this partition function is η-independent. Then we can integrate over η and obtain

$$Z = \sum_N \int D\bar{\Psi}\ D\Psi D A_\mu \delta(\partial_\mu j_\mu) \exp\{-\int d^2x\ [\bar{\Psi} i\partial\!\!\!/ \Psi + j_\mu A_\mu]\} \tag{7}$$

Now we represent $\delta(\partial_\mu j_\mu)$ in the form

$$\delta(\partial_\mu j_\mu) = \int DB_\mu D\theta \exp\{-\int d^2x \; [B_\mu j_\mu + \frac{1}{\pi} B_\mu \epsilon_{\mu\nu} \partial_\nu \theta]\} \tag{8}$$

Note that there is no factor of i in the exponent of the above expression just because we are representing the delta functional in Euclidean space. Introducing this expression in the generating functional and making the shift $B'_\mu = B_\mu + a_\mu$, which Jacobian is equal to one, we obtain

$$\begin{aligned} Z &= \sum_N \int D\bar{\Psi} \; D\Psi DA_\mu DB'_\mu D\theta \exp\{-\int d^2x \; [\bar{\Psi} i\partial\!\!\!/ \Psi + j_\mu (A_\mu^{c(N)} + B'_\mu) + \\ &+ \frac{1}{\pi} B'_\mu \epsilon_{\mu\nu} \partial_\nu \theta\} - \frac{1}{\pi} a_\mu \epsilon_{\mu\nu} \partial_\nu \theta]\} \end{aligned} \tag{9}$$

In order to factorize out a constrained fermionic model we have to eliminate the linear term in B'_μ. To this end we make a chiral transformation with parameter σ in the fermionic variables. This change yields a Fujikawa Jacobian given by [5]:

$$J_F = \exp\{\frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu \sigma (B'_\mu + A_\mu^{c(N)}) - \frac{1}{2\pi} \sigma \Box \sigma\} \tag{10}$$

The generating functional becomes:

$$Z = \sum_N \int D\bar{\Psi} \; D\Psi DA_\mu DB'_\mu D\theta e^{-S} \tag{11}$$

where

$$\begin{aligned} S &= \int d^2x \{\bar{\Psi} \; i\partial\!\!\!/ \Psi + (B'_\mu + A_\mu^{c(N)} + \epsilon_{\mu\nu} \partial_\nu \sigma) j_\mu + \frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu \theta B'_\mu - \frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu \theta a_\mu \\ &- \frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu \sigma A_\mu^{c(N)} - \frac{1}{\pi} \epsilon_{\mu\nu} \partial_\nu \sigma B'_\mu + \frac{1}{2\pi} \sigma \Box \sigma\} \end{aligned} \tag{12}$$

Now choosing $\sigma = \theta$, we cancel the linear term in B'_μ and obtain

$$Z = \sum_N \int D\bar{\Psi} \; D\Psi DA_\mu DB'_\mu D\sigma e^{-S'} \tag{13}$$

with

$$S = \int d^2x \{ \bar{\Psi}\ i\partial\!\!\!/ \Psi + (B'_\mu + A_\mu^{c(N)} + \epsilon_{\mu\nu}\partial_\nu\sigma) j_\mu - \frac{1}{\pi}\epsilon_{\mu\nu}\partial_\nu\sigma A_\mu + \frac{1}{2\pi}\sigma\Box\sigma \} \quad (14)$$

In order to write this generating functional as the product of a $\frac{U(1)}{U(1)}$ fermionic coset model and a bosonic part, like in Ref.[1], we make the shift $B'_\mu \to B'_\mu - \epsilon_{\mu\nu}\partial_\nu\sigma$. Thus we have

$$Z = \sum_N \int D\bar{\Psi}\ D\Psi D B'_\mu e^{-S_{cf}^{(N)}} \int D\sigma D A_\mu e^{-S_{bos}[\sigma, A_\mu^{(N)}]} \quad (15)$$

where

$$S_{cf}^{(N)} = \int d^2x \bar{\Psi}\ [i\partial\!\!\!/ + B\!\!\!/' + A\!\!\!/^{c(N)}]\Psi \quad (16)$$

and

$$S_{bos}[\sigma, A_\mu] = \int d^2x \frac{1}{\pi}[\sigma\frac{\Box}{2}\sigma - \epsilon_{\mu\nu}\partial_\nu\sigma A_\mu] \quad (17)$$

Let us stress that in our case both the $\frac{U(1)}{U(1)}$ fermionic coset factor and the bosonic one have a non-trivial topological structure.

If we want to identify this result with the one obtained in Ref.[3] we have to make a second chiral transformation in the fermionic variables, in order to decouple the B_μ field from the fermions [2]. In this way we recover a factorized generating functional $Z = Z_{Fer} \times Z_{Bos}$, with Z_{Fer} having the same zero modes problem which was studied in Ref.[3].

Calling the parameter of the transformation ϕ, we obtain:

$$\begin{aligned} Z &= \sum_N \int D\bar{\chi}\ D\chi \exp\{-\int d^2x\ [\bar{\chi}(i\partial\!\!\!/ + A\!\!\!/_c^N)\chi]\} \times \\ \int & DA_\mu D\sigma D\phi \exp\{\frac{1}{\pi}\int d^2x[\epsilon_{\mu\nu}\partial_\nu\sigma A_\mu - \frac{1}{2}\sigma\Box\sigma + \epsilon_{\mu\nu}\partial_\nu\phi A_\mu^c + \frac{1}{2}\phi\Box\phi]\} \\ &= Z_{Fer} \times Z_{Bos} \end{aligned} \quad (18)$$

From this expression, one can conclude that only the $N = 0$ sector contributes to the Z, a well established result for massless fermions [3].

Now, in order to compute fermionic correlation functions we have to add source terms in the form:

$$Z_{\rho_R,\rho_L} = \int D\bar{\Psi}\ D\Psi DA_\mu \exp\{-\int d^2x\ [\bar{\Psi} i\partial\!\!\!/ \Psi - j_\mu A_\mu + \rho_R \bar{\Psi}_R \Psi_R + \rho_L \bar{\Psi}_L \Psi_L]\} \tag{19}$$

so that, differentiating n_R times (n_L) with respect to ρ_R (ρ_L) we obtain the minimal correlation function in the n_R (n_L) sector, and making the cross derivatives n_R times with respect to ρ_R and n_L times with respect to ρ_L we arrive to the non-minimal correlator in the $\mid n_R - n_L \mid$ topological sector.

Let us consider the minimal functions in the $n_R > 0$ sector (equivalent results are obtained in the $n_L < 0$ sector) as follows:

$$< \prod_{i=0}^{n_R-1} \bar{\Psi}_R \Psi_R(x_i) >= Z^{-1} \frac{\delta^{n_R} Z}{\delta\rho_R(x_0)....\delta\rho_R(x_{n_R-1})} \tag{20}$$

This expression can be factorized as the product of a fermionic mean value and two bosonic parts in the form

$$\begin{aligned} < \prod_{i=0}^{n_R-1} \bar{\Psi}_R \Psi_R(x_i) > &= < \prod_{i=0}^{n_R-1} \bar{\chi}_R \chi_R(x_i) >_{fer} < e^{\sum_{i=0}^{n_R-1} \beta_i \phi(x_i)} >_{bos} \\ & < e^{\sum_{i=0}^{n_R-1} \beta_i \sigma(x_i)} >_{bos} \end{aligned} \tag{21}$$

where $\beta_i = 2$ for all values of i, and

$$L_F = \bar{\Psi}\ [i\partial\!\!\!/ + A\!\!\!/^{c(N)}]\Psi \tag{22}$$

$$L_\sigma = \frac{1}{\pi}[\sigma \frac{\Box}{2} \sigma - \epsilon_{\mu\nu} \partial_\nu \sigma A_\mu] \tag{23}$$

and

$$L_\phi = -\frac{1}{\pi}[\phi \frac{\Box}{2} \phi + \epsilon_{\mu\nu} \partial_\nu \phi A_\mu] \tag{24}$$

For simplicity we shall consider a distribution of the topological charge given by

$$\epsilon_{\mu\nu} \partial_\mu A_\nu^C = \sum_{i=0}^{n_R-1} \alpha_i(x) \delta^2(x - x_i) \tag{25}$$

with

$$\sum_{i=0}^{n_R-1} \alpha_i(x_i) = -2\pi N \tag{26}$$

This distribution corresponds to a Nielsen-Olesen configuration [6], in which the vortex width tends to zero (see Ref.[7] for details).

For the fermionic part we obtain

$$F(x_0, ..., x_{n_R-1}) = det_0(i\partial\!\!\!/ + A\!\!\!/_c^N) e^{\sum_{i=0}^{n_R-1} \alpha_i \beta_j \Box^{-1}_{x_i,x_j}} \prod_{i<j} \mid x_i - x_j \mid^2 \tag{27}$$

while the bosonic piece (i.e. the product of the last two factors in (16)) is given by

$$B(x_0, ..., x_{n_R-1}) = e^{-\sum_{i,j} (\frac{\alpha_i \alpha_j}{2\pi} + \alpha_i \beta_j + \frac{\pi}{2} \beta_i \beta_j) \Box^{-1}_{x_i,x_j}} \tag{28}$$

So, the minimal correlation function for the $n_R > 0$ topological sector reads:

$$< \prod_{i=0}^{N-1} \bar{\Psi}_R \Psi_R(x_i) >= det_0(i\partial\!\!\!/ + A\!\!\!/_c^N) \prod_{i,j=0 i\neq j}^{n_R-1} (\mid x_i - x_j \mid)^{\frac{-\alpha_i \alpha_j}{(2\pi)^2}} \tag{29}$$

This result is in full agreement with the one previously obtained in [3].

The computation of non-minimal functions is, though tedious, straightforward. The interested reader will find guiding lines in [7].

To conclude this letter, as promised in the introductory paragraph, we now go back to trivial topology and consider massive fermions. To be more specific let us examine the following generating functional:

$$Z = \int D\bar{\Psi}\ D\Psi \exp\{-\int d^2x\ [\bar{\Psi}(i\partial\!\!\!/ + m + A\!\!\!/)\Psi]\} \tag{30}$$

Following the steps of the previous case we make a gauge transformation in the fermionic variables and represent $\delta(\partial_\mu j_\mu)$ as in (8). Calling $B'_\mu = B_\mu + A_\mu$ we arrive at

$$
\begin{aligned}
Z &= \int D\bar{\Psi}\ D\Psi DB'_\mu D\theta \exp\{-\int d^2x\ [\bar{\Psi}i\partial\!\!\!/\Psi + j_\mu B'_\mu + m\bar{\Psi}\Psi \\
&+ \frac{1}{\pi}B'_\mu\epsilon_{\mu\nu}\partial_\nu\theta - \frac{1}{\pi}A_\mu\epsilon_{\mu\nu}\partial_\nu\theta]\} \qquad (31)
\end{aligned}
$$

We make now a chiral change with parameter σ in the fermionic variables:

$$
\begin{aligned}
Z &= \int D\bar{\Psi}\ D\Psi DB'_\mu D\theta \exp\{-\int d^2x\ [\bar{\Psi}i\partial\!\!\!/\Psi + j_\mu B'_\mu + me^{2\gamma_5\sigma}\bar{\Psi}\Psi \\
&+ \frac{1}{\pi}B'_\mu\epsilon_{\mu\nu}\partial_\nu\theta - \frac{1}{\pi}A_\mu\epsilon_{\mu\nu}\partial_\nu\theta - \frac{1}{\pi}B'_\mu\epsilon_{\mu\nu}\partial_\nu\sigma + \frac{1}{2\pi}\sigma\Box\sigma]\} \qquad (32)
\end{aligned}
$$

Again in order to eliminate the linear term in B'_μ we identify $\sigma = \theta$. We then find

$$
\begin{aligned}
Z[A_\mu] &= \int D\bar{\chi}\ D\chi DB'_\mu D\sigma_\mu \exp\{-\int d^2x[\bar{\chi}\ (i\partial\!\!\!/ + me^{2\gamma_5\sigma} + B\!\!\!/')\chi + \\
&+ \frac{1}{\pi}(\epsilon_{\mu\nu}\partial_\nu\sigma A_\mu + \frac{1}{2}\sigma\Box\sigma)]\} \qquad (33)
\end{aligned}
$$

At this point we shall make a perturbative expansion in the fermionic mass m. Obviously, the building block of this expansion is the following object:

$$
\bar{\chi}(x)\ e^{2\gamma_5\sigma(x)}\chi(x) = \chi_2^\dagger\ e^{2\sigma(x)}\chi_1 + \chi_1^\dagger\ e^{-2\sigma(x)}\chi_2 \qquad (34)
$$

Using this explicit expression, $Z_m[A_\mu]$ can be readily written in terms of fermionic and bosonic v.e.v's in the form:

$$
\begin{aligned}
Z_m[A_\mu] &= Z^{Fer}_{m=0}[coset]Z[A_\mu]^{Bos} \\
&\textstyle\sum_{j=0}^\infty \ \frac{m^{2j}}{(j!)^2}\int\prod_{k=1}^{j} d^2x_k d^2y_k < \chi_2^\dagger\ \chi_1(x_k)\chi_1^\dagger\ \chi_2(y_k) >_{Fer.Coset} \\
< &\textstyle\prod_{k=1}^j \ e^{2[\sigma(x_k)-\sigma(y_k)]} >_{Bos.\sigma,A_\mu} \qquad (35)
\end{aligned}
$$

Then, exactly as it happens in the massless case (Eq.(2)) the fermionic coset partition function can be extracted as an overall factor, but no complete factorization takes place, since every term in the mass expansion contains a fermionic v.e.v.. However, if one sets $A_\mu = 0$ the r.h.s. of Eq.(30) becomes the partition function of a bosonic Sine-Gordon model, as expected (Ref.[4],[8]). This fact can be easily verified just by evaluating the fermionic factor in the series, which can be done in a simple way using, for instance, the standard decoupling technique (See Ref.[2]).

In summary we have extended the factored coset approach to bosonization proposed in (Ref.[1]), in two directions. Firstly we discussed the case of massless fermions coupled to an Abelian gauge field with non-zero topological charge. We were able to show that the coset factorization takes place exactly as in the $N = 0$ case, but with both Z_{Fer} and Z_{Bos} containing the topological structure, i.e. the corresponding actions are dependent on N. Finally we considered the case $N = 0$ but with massive fermions. As it is well-known, the massive determinant cannot be exactly solved due to the chiral non-invariance of the mass term. This fact led us to make a perturbative expansion in the mass. Concerning this case our main conclusion is that no complete factorization is obtained because the constrained fermionic action enters the game through the v.e.v.'s which are present in the perturbative series of eq.(35).

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